EXISTENCE OF A σ FINITE INVARIANT MEASURE FOR A MARKOV PROCESS ON A LOCALLY COMPACT SPACE([1])

BY

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ABSTRACT

A σ finite invariant measure is found, for a Markov process, on a locally compact space, which maps continuous functions to continuous functions.

1. **Notation.** Let X be a locally compact Hausdorff space. Let $\sum$ denote its Baire sets. Let P be a Markov transition function on $(X, \sum)$ i.e.;

1.1 *For each* $x \in X, P(x; \cdot)$ *is a probability measure on* Σ *and for each* and $A \in \Sigma$, $P(\cdot\,; A)$ *is* Σ *measurable.*

The transition function defines an operator on bounded measurable functions and on measures by

1.2 $$(Pf)(x) = \int f(y)\, P(x, dy).$$

1.3 $$(\mu P)(A) = \int P(x, A)\, \mu(dx).$$

Let us denote

1.4 $$(T_\alpha f)(x) = \alpha(x) f(x).$$

1.5 $$T_A = T_\alpha \text{ where } \alpha = 1_A.$$

2. **The operator** $T_A P T_A$. Let A be the complements of the compact set B. Throughout the paper we shall assume

ASSUMPTION 2.1. $(T_A P T_A)^n 1(x) \to 0_{n\to\infty}$ *for every* $x \in X$.

REMARK. Assumption 2.1. is used here instead of the recurrence condition of Harris [1].

LEMMA 1. *Under Assumption* 2.1.

$$(PT_A)^n 1(x)_{n\to\infty} \to 0 \text{ for every } x \in X.$$

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Proof. $PT_A = T_APT_A + T_BPT_A$, where $(T_BPT_A)^2 = (T_APT_A)(T_BPT_A) = 0$. Hence $(PT_A)^n 1 = (T_APT_A)^n 1 + (T_BP)(T_APT_A)^{n-1} 1$ it is easy to see that if $(T_APT_A)^n 1$ tends to zero so does $(T_BP)(T_APT_A)^n 1$; note that $(T_APT_A)^{n+1} 1 = (T_APT_A)^n[(T_APT_A)1] \leqq (T_APT_A)^n 1$.

LEMMA 2. *Under Assumption* 2.1.

$$\sum_{n=1}^{\infty} (P^n 1_B)(x) > 0 \textit{ for every } x \in X.$$

Proof. Assume, to the contrary, that for some x_0 $(P^n 1_B)(x_0) = 0$ $n = 1,2,\cdots$. Now $(PT_A)^n 1 = [P(1 - T_B)]^n 1$ which is equal to 1 plus terms of the form $[\prod P(T_B)^{\varepsilon_i}]P1_B$ where ε_i is either zero or one. Thus at x_0 each of these terms is smaller than $(P^k 1_B)(x_0) = 0$ where $k \leqq n$. Thus $(PT_A)^n 1(x_0) = 1$ which violates Lemma 1.

REMARK. The conclusion of Lemma 2 is weaker than the assumption used by Nelson [2, Theorem 2.1.]. If Assumption 2.1. does not hold then $(T_APT_A)^n 1$ is a monotonically decreasing sequence whose limit g satisfies $T_APT_Ag = g$. Conversely if $(T_APT_A)^n 1 \to 0$ then no such invariant function exists:

$$|g(x)| = \lim |(T_APT_A)^n g(x)| \leqq \sup |g(y)| \lim_{n\to\infty} (T_APT_A)^n 1(x) = 0$$

Let X be the nonnegative integers and $P_{01} = 1$ $P_{n0} = 1 - \delta_n$ $P_{nn+1} = \delta_n$ where $0 < \delta_n < 1$ and $\sum \delta_n < \infty$. (This example is given in [2, p. 674]).

Let $B = \{0\}$. Then

$$T_APT_A = \begin{bmatrix} 0 & \delta_1 & 0 & \cdots \\ 0 & 0 & \delta_2 & \cdots \\ \vdots & \vdots & & \end{bmatrix}$$

and $f_n = (\delta_1 \cdots \delta_n)^{-1}$ is an invariant function for T_APT_A.

Thus Lemma 2 may be true and Assumption 2.1. false.

3. **The construction of the invariant measure.** Using the same notation as 1. Let β be a continuous function with:

3.1. $0 \leqq \beta \leqq 1$, $\beta = 1$ on B, $\beta = 0$ *outside* $\tilde{B}$ *where* $\tilde{B}$ *is compact.*

Put $\alpha = 1 - \beta$ then α is continuous and

3.2. $$0 \leqq \alpha \leqq 1 \quad \alpha \leqq 1_A.$$

Thus $T_Af \geqq 1_\alpha f$ for every positive function f and by Lemma 1.

3.3. $$(PT_\alpha)^n 1(x)_{n\to\infty} \to 0 \quad \textit{at every point } x.$$

Following Harris [1] let us define

3.4. $$P_N = \sum_{n=0}^{N} (PT_\alpha)^n PT_\beta$$

where P_N is an operator on bounded measurable functions defined on $\tilde{B}$.

Throughout the rest of this paper we shall assume

3.5. *Assumption. If f is continuous so is Pf.*

The operator P_N has the following properties

3.6. *If f is continuous then so is $P_N f$.*

3.7. *If $f \geqq 0$ then $P_N f \geqq 0$.*

3.8. $$P_N 1_{\tilde{B}} = \sum_{n=0}^{N} (PT_\alpha)^n P\beta = \sum_{n=0}^{A} (PT_\alpha)^n (1 - P\alpha)$$
$$= P1 - (PT_\alpha)^{N+1} 1 \leqq 1$$

Also if $f \geqq 0$ then

3.9. $P_{N+1} f \geqq P_N f$.

LEMMA 3. *Assume Assumptions 2.1 and 3.5. The sequence of operators P_N on $C(\tilde{B})$ converges uniformly. Let us denote its limit by P_∞. Then:*

(a) *If f is continuous on $\tilde{B}$ then so is $P_\infty f$.*

(b) *If $f \geqq 0$ then $P_\infty f \geqq 0$.*

(c) $P_\infty 1 = 1$.

Proof. Let us prove (c) first: $P_\infty 1 = \lim P_N 1 = 1 - \lim (PT_\alpha)^{N+1} 1 = 1$ by 3.3. Now $PT_\alpha 1 \leqq 1$ hence $(PT_\alpha)^N 1$ is monotonically decreasing to zero on $\tilde{B}$ hence converges uniformly there. Thus if $0 \leqq f \leqq M$ then

$$\| P_{N+K} f - P_N f \| = \left\| \sum_{n=N+1}^{N+K} (PT_\alpha)^n PT_\beta f \right\|$$
$$\leqq M \, \| P_{N+K} 1 - P_N 1 \|_{N \to \infty} \to 0.$$

This and 3.5 prove (a) and (b) follows from 3·6.

COROLLARY. *There exists a probability measure μ, on $\tilde{B}$, with $\mu P_\infty = \mu$.*

Proof. This is a standard argument: The collection of all probability measures, on $\tilde{B}$ is a bounded set of functionals on $C(\tilde{B})$ which is weakly closed and convex and invariant under P_∞ hence this set contains a fixed point.

Following Harris [1] let us define

(3.10. $$\lambda = \sum_{n=0}^{\infty} \mu (PT_\alpha)^n .$$

THEOREM. *Assume* 2.1. *and* 3.5. *The measure* λ *is* σ *finite, invariant under* P *and agrees with* μ *on* $\tilde{B}$ *(and thus not trivial).*

Proof. Let us first show that λ is σ finite Let

$$H = \{f: f \geqq 0 \text{ and } \langle \lambda, f \rangle = \int f d\lambda < \infty\}.$$

Let us show that $P^k\beta \in H$ for every $k \geqq 1$.

Now if $k = 1$

$$\langle \Sigma\ \mu(PT_\alpha)^n,\ P\beta \rangle = \langle \Sigma\ \mu(PT_\alpha)^n,\ 1 - (PT_\alpha)1 \rangle = \langle \mu, 1 \rangle = 1.$$

Assume $P^k\beta \in H$ then

$$\begin{aligned}\langle \Sigma\ \mu(PT_\alpha)^n, P^{k+1}\beta \rangle &= \langle \Sigma\ \mu(PT_\alpha)^n, PT_\alpha[P^k\beta] \rangle \\ &\quad + \langle \Sigma\ \mu(PT_\alpha)^n,\ PT_\beta[P^k\beta] \rangle \\ &\leqq \langle \Sigma\ \mu(PT_\alpha)^{n+1}, P^k\beta \rangle + M \langle \Sigma\ \mu(PT_\alpha)^n, PT_\beta 1 \rangle\end{aligned}$$

where $M = \sup(P^k\beta)(y)$.

The first term is finite by the induction hypothesis and the second term is $M \langle \Sigma\ \mu(PT_\alpha)^n,\ P\beta \rangle = M$. Thus λ is σ finite on

$$\bigcup_{k=1}^{\infty} \{x : (P^k\beta)(x) > 0\} \supset \bigcup_{k=1}^{\infty} \{x : (P^k 1_B)(x) > 0\}$$

by 3.1 and by Lemma 2 λ is σ finite on all of X.

Now if f is supported by $\tilde{B}$ then $(PT_\alpha)^n f = 0$ $n = 1$ $2, \cdots$ and $\langle \lambda, f \rangle = \langle \mu, f \rangle$.

Finally

$$\begin{aligned}\lambda P &= \Sigma\ \mu(PT_\alpha)^n P = \Sigma\ \mu(PT_\alpha)^n PT_\alpha + \Sigma \mu(PT_\alpha)^n PT_\beta \\ &= \lambda - \mu + \Sigma\ \mu(PT_\alpha)^n PT_\beta\end{aligned}$$

but $\mu = \Sigma\ \mu(PT_\alpha)^n PT_\beta$ hence $\lambda P = \lambda$.

REMARK Any compact set is covered by a finite union of sets of the form $\{x : (P^q\beta)(x) > 1/n\}$ and thus λ is finite on compact sets.

REFERENCES

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